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On the Norms of Metric Projections

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It is shown that a Banach space is uniformly non-square if and only if the supremum of the norms of the metric projections onto proximinal subspaces is less than 2; it is also shown that a Banach space of dimension at least 3 is a Hilbert space if and only if the same supremum is equal to 1. Some examples are given of Banach spaces that contain Chebyshev subspaces whose metric projections are linear and have norm 2.

1. INTRODUCTION

Let X be a real Banach space and M a non-trivial, closed, proper subspace of X. The *metric projection* onto M is the mapping $P_M: X \to 2^M$, which associates with each x in X its (possibly void) set of best approximations in M; that is,

$$P_M(x) = \{m \in M: \|x - m\| = d(x, M)\},$$

where $d(x, M) = \inf\{\|x - y\|: y \in M\}$. The subspace M is called *proximinal* (resp. *Chebyshev*) if $P_M(x)$ contains at least (resp. exactly) one point for each x in X. If M is a proximinal subspace of X, define the *norm* of P_M by

$$\|P_M\| = \sup\{\|b\|: b \in P_M(x) \text{ and } \|x\| \leqslant 1\}.$$

Note this definition extends the one given in [2] where M is only allowed to be a Chebyshev subspace of X. Since $P_M(m) = \{m\}$ for m in M, it follows that $\|P_M\| \geqslant 1$. If b is in $P_M(x)$ for x in X, then

$$\|b\| \leqslant \|b - x\| + \|x\| \leqslant 2\,\|x\|$$

and hence $\|P_M\| \leqslant 2$. Thus $1 \leqslant \|P_M\| \leqslant 2$ for every proximinal subspace M of X.

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It is well known that in a Hilbert space the metric projection onto a closed subspace is the orthogonal projection and hence is single-valued, linear, and has norm 1. Deutsch and Lambert [2] proved that for each real number r with $1 \leqslant r < 2$, there is a Chebyshev subspace M of $l^1(2)$, the two dimensional l^1 space, such that P_M is linear and has norm r. They also showed that $C[0, 1]$ contains a Chebyshev subspace whose metric projection is linear and has norm 2. Thus metric projections (in fact, even linear ones) onto Chebyshev subspaces exist with every norm size possible.

The purpose of this paper is to study the relationship between the geometry of a given Banach space and the norms of the metric projections onto its proximinal subspaces. This investigation was motivated by a question posed by Deutsch and Lambert [2], asking exactly which Banach spaces contain Chebyshev subspaces having linear metric projections with norm 2. While the results of this paper shed some light on this question, the spotlight here is on the supremum of the norms of the metric projections onto proximinal subspaces.

Every Banach space contains proximinal subspaces, for example, finite dimensional subspaces and null spaces of continuous linear functionals that attain their norm on the unit ball. It is well known that every subspace of a Banach space X is a proximinal subspace if and only if X is reflexive.

For a Banach space X, define the *metric projection bound* of X, denoted $\mathrm{MPB}(X)$, by

$$\mathrm{MPB}(X) = \sup\{\|P_M\| : M \text{ is a proximinal subspace of } X\}.$$

From the discussion above, for every space X, it follows that $1 \leqslant \mathrm{MPB}(X) \leqslant 2$. In particular, if H is a Hilbert space, then $\mathrm{MPB}(H) = 1$; also, $\mathrm{MPB}(l^1(2)) = 2$ and $\mathrm{MPB}(C[0, 1]) = 2$.

Section 2 contains the main results of this paper. First, a Banach space X is uniformly non-square if and only if $\mathrm{MPB}(X) < 2$. Using this characterization, it is shown that if X is not uniformly non-square, then the space $l^2(X)$ contains a proximinal subspace M with $\|P_M\| = 2$. Second, a Banach space X of dimension at least 3 is a Hilbert space if and only if $\mathrm{MPB}(X) = 1$.

In Section 3, an example is given to show that it is possible for X to contain a Chebyshev subspace whose metric projection is linear and has norm 2 even though X has several nice rotundity and smoothness properties. The paper concludes with some remarks relevant to the Deutsch and Lambert question.

2. The Metric Projection Bound

A Banach space X is *uniformly non-square* if there is a positive number δ such that there do not exist elements x and y of the unit ball for which

$\|\frac{1}{2}(x+y)\| > 1-\delta$ and $\|\frac{1}{2}(x-y)\| > 1-\delta$. James [6] introduced this notion and proved that uniformly non-square spaces are reflexive. It is now known, from the work of James and Enflo, that a Banach space is superreflexive if and only if it has an equivalent uniformly non-square norm (see [1, p. 169]).

THEOREM 2.1. *A Banach space X is uniformly non-square if and only if* $\mathrm{MPB}(X) < 2$.

Proof. Assume X is uniformly non-square and let δ be the positive number guaranteed in the definition. Suppose there is a proximinal subspace M of X such that $\|P_M\| > 2-\delta$. Choose an element x of unit norm and b in $P_M(x)$ such that $\|b\| > 2-\delta$ and let $y = b-x$. Then $\|y\| \leqslant \|x\| = 1$ and $\|\frac{1}{2}(x+y)\| > 1-\delta/2$. Now, from the triangle inequality, it follows that $\|y\| > 1-\delta$ and hence

$$\|\tfrac{1}{2}(x-y)\| = \|x-\tfrac{1}{2}b\| \geqslant \|y\| > 1-\delta.$$

The existence of x and y contradict the hypothesis on X. Thus $\mathrm{MPB}(X) \leqslant 2-\delta$.

For the converse, assume X is not uniformly non-square. Let $\varepsilon > 0$ be given. It must be shown that there exists a proximinal subspace M of X such that $\|P_M\| > 2-\varepsilon$. Choose a positive integer n such that $10^{-n} < \varepsilon/4$. Since X is not uniformly non-square, there exist elements x and y of unit norm in X such that the unit sphere S of the two dimensional subspace of X determined by x and y lies inside the parallelogram generated by $\pm(1+10^{-n}\varepsilon/4)x$ and $\pm(1+10^{-n}\varepsilon/4)y$. For convenience, let P denote the parallelogram generated by $\pm x$ and $\pm y$. Thus S lies inside $(1+10^{-n}\varepsilon/4)P$. Let M be the one dimensional subspace of X generated by $u = x+(1-\varepsilon/4)y$. Note that M is a proximinal subspace of X. To show $\|P_M\| > 2-\varepsilon$ it suffices to show there exists b in $P_M(x)$ such that $\|b\| > 2-\varepsilon$. Since $x+(1-\varepsilon/4)S$ intersects M at u (and possibly some other points), it follows that $d(x, M) \leqslant 1-\varepsilon/4$. From this, the fact that S lies inside $(1+10^{-n}\varepsilon/4)P$, and the observation that $(1-\varepsilon/4)(1+10^{-n}\varepsilon/4) < 1-(1-10^{-n})\varepsilon/4$, it follows that $x+d(x, M)S$ lies inside the parallelogram $P_1 \equiv x+(1-(1-10^{-n})\varepsilon/4)P$. Therefore every element of $P_M(x)$ lies inside P_1. Let z and w be the points of intersection of M with P_1 labeled in such a way that $\|z\| < \|w\|$. Let b be any element of $P_M(x)$. Then b lies in M between z and w and hence $\|b\| \geqslant \|z\|$. Thus it suffices to show that $\|z\| > 2-\varepsilon$. Let $P_0 \equiv x+P$, let $P_2 \equiv x+(1-\varepsilon/4)P$, and let $t = x+(1-(1-10^{-n})\varepsilon/4)y$. Observe that the parallelograms P_0, P_1, and P_2 are all multiples of P translated by x and that the one dimensional subspace M intersects P_0, P_1, and P_2 at the points 0, z, and u, respectively. Since $\|u-t\| = 10^{-n}\|u-(x+y)\|$, it follows that $\|u-z\| = 10^{-n}\|u-0\|$. But $\|u\| \leqslant 2$ and hence $\|u-z\| \leqslant \varepsilon/2$ by the choice of n. Thus

$$\|x+y-z\| \leqslant \|x+y-u\| + \|u-z\| \leqslant 3\varepsilon/4. \qquad (*)$$

Since S lies inside $(1+10^{-n}\varepsilon/4)P$, it follows that $\|x+y\| > 2/(1+10^{-n}\varepsilon/4) > 2-\varepsilon/4$. From this inequality and inequality (*), using the triangle inequality, it follows that $\|z\| > 2-\varepsilon$. This completes the proof.

In the following corollaries, X^* denotes the dual space of the Banach space X. The first two corollaries follow easily since X is uniformly non-square if and only if X^* is uniformly non-square (see [1, p. 173]) and since X is uniformly non-square if it is either uniformly rotund or uniformly smooth (see [1, Chap. VII, Sect. 2]).

COROLLARY 2.2. $\mathrm{MPB}(X) < 2$ *if and only if* $\mathrm{MPB}(X^*) < 2$.

COROLLARY 2.3. *If X is uniformly rotund or uniformly smooth, then* $\mathrm{MPB}(X) < 2$ *and* $\mathrm{MPB}(X^*) < 2$.

COROLLARY 2.4. *If X is not uniformly non-square and $1 \leqslant p < \infty$, then the Banach sequence space $l^p(X)$ contains a proximinal subspace M such that $\|P_M\| = 2$.*

Proof. By Theorem 2.1, $\mathrm{MPB}(X) = 2$ and hence there exists a sequence $\{M_i\}$ of proximinal subspaces of X such that $\|P_{M_i}\| \geqslant 2 - i^{-1}$. Let M be the subspace of $l^p(X)$ defined by $m = (m_i)$ belongs to M if and only if m_i belongs to M_i for each i. It is easy to verify that M is a proximinal subspace of $l^p(X)$ and, in fact, $P_M(x) = \{b = (b_i) : b_i \in P_{M_i}(x_i) \text{ for each } i\}$ for $x = (x_i)$ in $l^p(X)$. Also $\|P_M\| \geqslant \|P_{M_i}\|$ for each i and hence $\|P_M\| = 2$. This completes the proof.

As mentioned in the introduction, if X is a Hilbert space, then $\mathrm{MPB}(X) = 1$. It will be shown that the converse is also true whenever X is a Banach space of dimension at least 3.

Let $\perp$ denote James' orthogonality; that is, $x \perp y$ if and only if $\|x\| \leqslant \|x + \alpha y\|$ for all real numbers α. It is well known (see [5]) for a Banach space X of dimension at least 3 that $\perp$ is symmetric (that is, $x \perp y$ implies $y \perp x$) if and only if X is a Hilbert space.

THEOREM 2.5. *For X a Banach space,* $\mathrm{MPB}(X) = 1$ *if and only if $\perp$ is symmetric. Hence, whenever the dimension of X is at least* 3, $\mathrm{MPB}(X) = 1$ *if and only if X is a Hilbert space.*

Proof. Assume that $\perp$ is symmetric and M is a proximinal subspace of X. Since $x - P_M(x) \perp M$, it follows that $M \perp x - P_M(x)$ and, in particular, $P_M(x) \perp x - P_M(x)$. Therfore, for b in $P_M(x)$, it follows that $\|b\| \leqslant \|b + x - b\| = \|x\|$ and hence $\|P_M\| \leqslant 1$.

For the reverse implication, assume $\mathrm{MPB}(X) = 1$ and $x \perp y$. Let M be the one dimensional subspace of X generated by y. Since, for α a real number, $\alpha x \perp y$, it follows that y is an element of $P_M(\alpha x + y)$ and hence $\|y\| \leqslant \|\alpha x + y\|$ because $\mathrm{MPB}(X) = 1$. Therefore $y \perp x$. This completes the proof.

3. EXAMPLES AND REMARKS

EXAMPLE 3.1. Let X_1 be the product space $P_{l^2}(B_i)$, where B_i is the two dimensional l^{i+1} space (see [1, p. 35] for the definition). Since $\lim_{i\to\infty} \|(\alpha,\beta)\|_{l^{i+1}} = \max\{|\alpha|, |\beta|\}$, it follows, by the proof of Theorem 2.1, that for each positive integer n there exists $j = j(n)$ such that B_j contains a one dimensional subspace M_j with $\|P_{M_j}\| \geqslant 2 - n^{-1}$. Note that M_j is a Chebyshev subspace of B_j since B_j is rotund. Let M be the subspace of X_1 defined by $m = (m_i)$ belongs to M if and only if m_i belongs to M_j if $i = j(n)$ for some n and $m_i = 0$ otherwise. Then M is a Chebyshev subspace of X_1; in fact, for $x = (x_i)$ in X_1, it is easy to verify that $P_M(x) = (m_i)$, where $m_i = P_{M_j}(x_j)$ if $i = j(n)$ for some n and $m_i = 0$ otherwise. Now, $\|P_M\| \geqslant \|P_{M_j}\|$ for all j and hence $\|P_M\| = 2$. Also note that P_M is linear since P_{M_j} is linear for all j; that P_{M_j} is linear follows from the general fact that any Chebyshev hyperplane in a Banach space has linear metric projection (see [4, p. 159]). So X_1 has a proper Chebyshev subspace whose metric projection is linear and has norm 2 yet the norm on X_1 is locally uniformly rotund, weakly uniformly rotund, Fréchet differentiable, and uniformly Gateaux differentiable (see [1, Chap. VII, Sect. 2]). Also, note that X_1 is superreflexive; in fact, X_1 is isomorphic to a Hilbert space.

The method of proof of Corollary 2.4 and the method of constructing Example 3.1 provide a general technique to produce examples, the results of which are collected in the following statement.

PROPOSITION 3.2. *Let p be such that $1 \leqslant p < \infty$. If $\{X_i\}$ is a sequence of Banach spaces, then $P_{l^p}(X_i)$ contains a proximinal subspace M with $\|P_M\| \geqslant$* $\limsup_{i\to\infty} \mathrm{MPB}(X_i)$.

In particular, if X is a Banach space, then $l^p(X)$ contains a proximinal subspace M with $\|P_M\| \geqslant \mathrm{MPB}(X)$.

EXAMPLE 3.3. The space l^1 contains a Chebyshev subspace whose metric projection is linear and has norm two. This follows from the technique capsulated in Proposition 3.2 since l^1 is isometrically isomorphic to $l^1(l^1(2))$ and since for each i the space $l^1(2)$ contains a Chebyshev subspace M_i with $\|P_{M_i}\| \geqslant 2 - i^{-1}$ (see [2]); that M is Chebyshev and P_M is linear follow as in Example 3.1.

A general answer (at least an answer in terms of the geometry of Banach spaces) to the Deutsch and Lambert question seems elusive. By Lemma 4.3 of [2] and Theorem 2.1, a necessary condition for X to contain a Chebyshev subspace with metric projection of norm two is that X is infinite dimensional and not uniformly non-square. However, this condition is not sufficient since $l_0^\infty(\Gamma)$ with $\mathrm{card}(\Gamma) > c$ has no Chebyshev subspaces at all (see [3] or

[4, p. 117]). Determining a characterizing condition is difficult (if not impossible) since (1) the spaces $C[0, 1]$, l^1, and X_1 contain Chebyshev subspaces whose metric projections are linear and have norm two yet the geometry and structure of these spaces are so greatly different, and (2) given any Banach space X, there is a Banach space containing X as a complemented subspace which has a Chebyshev subspace whose metric projection is linear and has norm 2 (take, for example, $X \oplus_2 l^1$).

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